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APPLICATION OF BOUNDARY LAYER THEORY TO THE SOLUTION OF PROBLEMS WITH COUPLED HEAT AND MASS TRANSFER

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The possibility of applying the approximate integral method of boundary layer theory to the solution of problems with coupled heat and mass transfer is examined. Solutions are presented for a number of specific problems. These include cases involving variable physical properties.

Intensive development of irreversible thermodynamics has recently led to a quite rigorous theory for heat and mass transfer through porous media [1].

At the same time, linearization of the differential equations describing heat and mass transfer, which is valid for the so-called zonal treatment of volume and also for a narrow range of variation of temperature and moisture content, cannot, in general, be considered accurate enough. Moreover, the exact solutions that have been obtained for linear problems are, in a number of cases, very awkward in form and not amenable to analysis. Since exact solutions of nonlinear problems can scarcely be obtained in closed form, it is expedient to resort to approximate methods of solution, among which should be included the calculus of variations and the integral methods of the theory of the hydrodynamic boundary layer. Integral methods have been applied lately to the solution of nonlinear and unsteady problems with independent heat and mass transfer [2-6]. The basic principles are presented in the references cited.

We shall examine the possibility of applying the one-parameter integral method, equivalent to the Karman-Pohlhausen method in boundary layer theory, to the solution of problems with combined heat and mass transfer in capillary-porous media. In the plane one-dimensional case the equations describing heat and mass transfer for dimensionless transfer potentials have the form:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(A_1 \frac{\partial T}{\partial x} - A_2 \frac{\partial \Theta}{\partial x}\right), \tag{1}$$

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x}\left(A_3 \frac{\partial \Theta}{\partial x} - A_4 \frac{\partial T}{\partial x}\right). \tag{2}$$

We shall first illustrate the general method for the case of constant coefficients A_i:

$$\frac{\partial T}{\partial t} = A_1 \frac{\partial^2 T}{\partial x^2} - A_2 \frac{\partial^2 \Theta}{\partial x^2}, \tag{3}$$

$$\frac{\partial \Theta}{\partial t} = A_3 \frac{\partial^2 \Theta}{\partial x^2} - A_4 \frac{\partial^2 T}{\partial x^2}. \tag{4}$$

We shall examine the boundary problem:

$$T(0,\ t) = \Theta(0,\ t) = 1;\ T(x,\ 0) = \Theta(x,\ 0) = 1. \tag{5}$$

Consider a thermal boundary layer of thickness δ_T and a boundary layer of mass transfer potential of thickness δ_Θ, described by the condition that the potentials are respectively equal to their initial values outside each layer. As in the case of heat transfer to a laminar flow over a body, when the thermal and dynamic boundary layers are examined concurrently [7], we may distinguish three cases, characterized by the ratio of the thickness δ_Θ and δ_T : 1) $\mu = \delta_\Theta/\delta_T < 1$, 2) $\mu = 1$, and 3) $\mu > 1$. The condition $\mu = 1$ indicates that the heat and mass transfer potential fields are similar, the velocities of propagation of the T and Θ fronts being identical. For $\mu < 1$ the front of the heat transfer potential T leads that of the mass transfer potential Θ, and vice versa for $\mu > 1$. We shall examine separately the cases $\mu \leq 1$ and $\mu > 1$.

Case $\mu \geq 1$: Integrating (3) over the thickness δ_T, and (4) over the thickness δ_Θ and taking into account that when $\eta_\Theta > 1$ $\Theta = \Theta(x,\ 0) = 0$, and when $\eta_t > 1$ $T = T(x, 0) = 0$, we obtain

$$\frac{1}{3}\frac{d\delta_T}{dt}=\frac{2A_1}{\delta_T}-\frac{2A_2}{\mu\delta_T}, \qquad (6)$$

$$\frac{1}{3}\frac{d\delta_\Theta}{dt}=\frac{2A_3}{\delta_\Theta}-\frac{2A_4\mu^2}{\delta_\Theta}. \qquad (7)$$

It has been further assumed in deriving (6) and (7) that within each of the layers the potentials T and Θ may be approximated by means of quadratic polynomials satisfying the boundary and initial conditions and having the form:

$$T=(1-\eta_T)^2;\ \Theta=(1-\eta_\Theta)^2;\ \eta_T=\frac{x}{\delta_T};\quad \eta_\Theta=\frac{x}{\delta_\Theta}. \qquad (8)$$

Solution of these ordinary first-order differential equations, with account for the initial conditions $\delta_T(0)=0$ and $\delta_\Theta(0)=0$ gives:

$$\delta_T=\left[12\left(A_1t-A_2\int\limits_0^t\frac{dt}{\mu}\right)\right]^{1/2}, \qquad (9)$$

$$\delta_\Theta=\left[12\left(A_3t-A_4\int\limits_0^t\mu^2\,dt\right)\right]^{1/2}, \qquad (10)$$

Thus, we have the following relation for determining μ:

$$\mu^2=\left(A_3t-A_4\int\limits_0^t\mu^2\,dt\right)\Big/\left(A_1t-A_2\int\limits_0^t\frac{dt}{\mu}\right). \qquad (11)$$

The value of μ satisfying (11) is

$$\mu=\mu_0=\left(A_2\pm\sqrt{A_2^2+4A_3(A_1+A_4)}\right)\Big/2(A_1+A_4). \qquad (12)$$

In order to choose the sign of the root, we examine the limiting case of independent heat and mass transfer, corresponding to $Ko^* = Pn = 0$. In this case, from (9) and (10) $\delta_T=(12a_mt/\mathrm{Lu})^{1/2}$ and $\delta_\Theta=(12a_mt)^{1/2}$, which gives $\mu=\mathrm{Lu}^{1/2}$. Thus, the "minus" sign in front of the root in (12) should be discarded.

When certain appropriate constraints are satisfied, the initial condition $\mu\leq 1$ turns out to be equivalent to

$$\rho=\frac{A_2+A_3}{A_1+A_4}=\mathrm{Lu}\frac{1+\mathrm{Ko}^*}{1+\mathrm{Lu\,Pn}(1+\mathrm{Ko}^*)}\leqslant 1. \qquad (13)$$

Case $\mu > 1$: Carrying out calculations similar to the foregoing, we obtain

$$\delta_T=\left[12\left(A_1t-A_2\int\limits_0^t\frac{dt}{\mu^2}\right)\right]^{1/2}, \qquad (14)$$

$$\delta_\Theta=\left[12\left(A_3t-A_4\int\limits_0^t\mu\,dt\right)\right]^{1/2}, \qquad (15)$$

$$\mu^2=\left(A_3t-A_4\int\limits_0^t\mu\,dt\right)\Big/\left(A_1t-A_2\int\limits_0^t\frac{dt}{\mu^2}\right). \qquad (16)$$

Equation (16) is satisfied by

$$\mu = \mu_0 = \left(-A_4 + \sqrt{A_4^2 + 4A_1(A_2 + A_3)}\right) / 2A_1. \tag{17}$$

The condition $\mu > 1$ is equivalent to:

$$\rho = \mathrm{Lu}(1 + \mathrm{Ko}^*)/[1 + \mathrm{Lu}\,\mathrm{Pn}(1 + \mathrm{Ko}^*)] > 1. \tag{18}$$

Thus, the parameter ρ, which is a function of the parameters Lu, Pn, and Ko*, determines, in a manner similar to the Prandtl number in the theory of the dynamic boundary layer, the relationship between the values and the velocities of propagation of the thermal layer and the mass transfer potential layer, i.e., in the last analysis, it characterises the inertia of the heat and mass transfer field for nonstationary problems. Thus when $\rho > 1$ the mass potential field propagates faster than the temperature field, and the case $\rho < 1$ corresponds to faster propagation of the temperature field, and, finally, when $\rho = 1$ the two fields are similar. It is easy to see that when there are no changes of phase and no thermal gradient mass transfer (which is equivalent to no cross terms in the Onsager equations), the parameter ρ becomes the Lykov number Lu. The asymmetry of the potential fields when LU = 1, noted in [1], is explained by the additional effect of the Pn and Ko* numbers on the inertia of both fields.

The final solution of the above problem may be obtained by substituting values of μ from (12) or (17) followed by substitution in (9)-(10) or (14)-(15), respectively. The profiles of the potentials T and Θ will then be determined by the relations (8) with the appropriate values for δ_T and δ_Θ.

We shall now turn to the more general case in which the coefficients A_i depend on the heat and mass transfer potentials: $A_i = A_i(\Theta), T)$. The linear problem, with coefficients A_i depending on the coordinates and time, may be similarly examined. In this case we shall assign boundary conditions of the first kind, with given constant temperature at the boundary of the half-space and a constant mean value for the mass transfer potential inside the layer δ_Θ.

$$T(0,\ t) = 1;\ \int_0^1 \Theta\, d\eta_\Theta = 1. \tag{19}$$

The expressions for the approximate polynomials take the form:

$$T(x,\ t) = (1 - \eta_T)^2;\ \Theta(x,\ t) = 3(1 - \eta_\Theta)^2. \tag{20}$$

When $\mu \le 1$

$$\delta_T = [12 \int_0^t (A_1 + 3A_2/\mu)|_{x=0}\, dt]^{1/2}, \tag{21}$$

$$\delta_\Theta = 2\left\{\int_0^t [(\mu A_4 + 3A_3)|_{x=0} + (\mu^2 - 1)A_4|_{x=\delta_\Theta}]\, dt\right\}^{1/2}. \tag{22}$$

For determining μ we have the integral relation

$$\mu^2 = \frac{\int_0^t [(\mu A_4 + 3A_3)|_{x=0} + (\mu^2 - 1)A_4|_{x=\delta_\Theta}]\, dt}{3\int_0^t (A_1 + 3A_2/\mu)|_{x=0}\, dt}. \tag{23}$$

The value of μ may be determined for a specified form of the functional dependence $A_i(\Theta, T)$, for example, by the method of successive approximations. If all the A_i are assumed constant in the first approximation, then, to determine μ, we obtain as before a quadratic equation which may easily be solved.

When $\mu > 1$

$$\delta_T = \left\{12\int_0^t \left[(A_1 + 3A_2/\mu)|_{x=0} + \frac{3(1-\mu)}{\mu^2} A_2\Big|_{x=\delta_T}\right] dt\right\}^{1/2}, \tag{24}$$

$$\delta_\Theta = \left\{ 12 \int_0^t (A_3 + \mu A_4/3)|_{x=0} dt \right\}^{1/2} . \quad (25)$$

For determining μ we have an integral relation of the type

$$\mu^2 = \frac{\int_0^t (A_3 + \mu A_4/3)|_{x=0}\, dt}{\int_0^t \left[(A_1 + 3A_2/\mu)|_{x=0} + \frac{3(1-\mu)}{\mu^2} A_2 \Big|_{x=\delta_T} \right] dt} . \quad (26)$$

It is evident that when $\mu = 1$, Eqs. (23) and (26) coincide.

Since it is necessary, in obtaining the final results, to know the value of the potential Θ at the surface, we proceed as follows. Having assigned a definite form of the dependence $A_i(T, \Theta)$ and having determined the parameters μ, δ_T and δ_Θ as functions of $\Theta(0, t)$, we can find the unknown quantity $\Theta(0, t)$ from the second relation of (20), putting $x = 0$ and solving the resulting algebraic equation.

We note that the above-mentioned method is also applicable, without any serious limitations, to the investigation of heat and mass transfer processes in binary gas mixtures.

NOTATION

T, Θ — dimensionless potentials of heat and mass transfer; A_i — coefficients characterizing transfer of the corresponding entity under the action of the motive force; $A_1 = a_m \left(\frac{1}{\mathrm{Lu}} + \mathrm{Fe}\right)$, $A_2 = a_m \mathrm{Ko}^*$, $A_3 = a_m$, $A_4 = a_m \mathrm{Pn}$; a_m a_m — coefficient of mass transfer potential conduction Lu, Pn, Ko*, Fe — Lykov, Posnov, Kossovich, and Fedorov numbers, respectively; δ_T, δ_Θ — thermal layer and layer of mass transfer potential; $\mu = \delta_\theta/\delta_T$, $\eta_T = x/\delta_T$, $\eta_\theta = x/\delta_\theta$; ρ — modified inertia parameter of coupled heat and mass transfer processes; x — coordinate; t — time.

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